Final - Computer Science 2 (2021-22)
Time: 3 hours.
Attempt all questions, giving proper explanations.

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

   (a) Apply the first iteration of the *classical Jacobi method*: Find the orthogonal matrix $\mathbf{P}$ such that the $(3,4)$th entry of $\mathbf{A}^{(1)} := \mathbf{P}^T\mathbf{A}\mathbf{P}$ is 0, and write down $\mathbf{A}^{(1)}$. **[5 marks]**

   (b) For a matrix $\mathbf{M}$ let

$$L(\mathbf{M}) := \sum_{i \neq j} |m_{ij}|^2$$

   be the sum of squares of the off-diagonal entries of $\mathbf{M}$. If $\mathbf{A}^{(k)}$, $k \geq 0$ are the successive iterates of the classical Jacobi method, show that

$$L\left(\mathbf{A}^{(k+1)}\right) < L\left(\mathbf{A}^{(k)}\right). \qquad \textbf{[4 marks]}$$

2. Consider a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$ with $n \geq 2$. Consider the Gerschgorin discs

$$D_i = \{z \in \mathbb{C} : \ |z - a_{ii}| \leq R_i(\mathbf{A})\},$$

   where $R_i(\mathbf{A}) := \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$.

   (a) State Gerschgorin's first and second theorems. (no need to prove) **[4 marks]**

   (b) Now let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. *Assume* Gerschgorin's first and second theorems, and $\max_i R_i(\mathbf{A}^{(k)}) \xrightarrow{k \to \infty} 0$ for the iterates $\mathbf{A}^{(k)}$ in the classical Jacobi method. Let $b_1^{(k)} \leq b_2^{(k)} \leq \cdots \leq b_n^{(k)}$ be the ordering of the diagonal entries of $\mathbf{A}^{(k)}$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\mathbf{A}$. Argue that $(b_1^{(k)}, b_2^{(k)}, \cdots, b_n^{(k)}) \xrightarrow{k \to \infty} (\lambda_1, \lambda_2, \cdots, \lambda_n)$. **[4 marks]**

3. (a) Find the polynomial p_3 of degree 3 which passes through the points $(0,1)$, $(1,e)$, $(2,e^4)$, $(3,e^9)$. **[3 marks]**

   (b) Give the best possible bound on $\sup_{x \in [0,4]} |e^{x^2} - p_3(x)|$. **[3 marks]**

   (c) Find the Newton-Cotes approximation for $\int_0^3 e^{x^2}\,dx$ using the quadrature points $0, 1, 2, 3$. **[2 marks]**

   (d) Give the best possible bound on the error in the approximation in 3c. **[2 marks]**

4. Let $f : [0,1] \to \mathbf{R}$ be an infinitely differentiable function. The Hermite interpolation polynomial approximating a function f at the interpolation points $0 \leq x_0 < x_1 < \cdots < x_n \leq 1$ is a polynomial of degree $2n+1$ given by

$$p_{2n+1}(x) = \sum_{k=0}^{n} H_k(x) f(x_k) + \sum_{k=0}^{n} K_k(x) f'(x_k),$$

   where

$$H_k(x) = [L_k(x)]^2 \left[1 - 2L_k'(x_k) \cdot (x - x_k)\right],$$
$$K_k(x) = [L_k(x)]^2 (x - x_k).$$

   Here

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i}\right).$$

We showed in class that by choosing $x_0, x_1, \cdots x_n$ to be the distinct zeroes of ϕ_{n+1}, the $(n+1)$th orthogonal polynomial in $[0,1]$, one obtains

$$\int_0^1 p_{2n+1}(x)dx = \sum_{k=0}^{n} W_k f(x_k),$$

where

$$W_k = \int_0^1 [L_k(x)]^2 \, dx.$$

(a) Show that for this choice of $x_0 < x_1 < \cdots < x_n$ we have

$$\int_0^1 [L_k(x)]^2 \, dx = \int_0^1 [L_k(x)] \, dx \qquad \text{for each } k. \qquad \textbf{[5 marks]}$$

(b) Show that for this choice of $x_0 < x_1 < \cdots < x_n$ we have

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} W_k f(x_k)$$

whenever $f \in \mathscr{P}_{2n+1}$ (the collection of polynomials of degree at most $2n+1$). **[2 marks]**

(c) Let q_n be the Lagrange polynomial of degree n approximating f at the interpolation points $x_0, x_1, \ldots, x_n$ chosen above. Show that

$$\int_0^1 f(x)dx = \int_0^1 q_n(x)dx$$

whenever $f \in \mathscr{P}_{2n+1}$. **[3 marks]**

(d) For $n = 1$ find x_0, x_1. **[4 marks]**

(e) Explain why the Gauss quadrature rule for $n = 1$ is better than the trapezium rule. **[2 marks]**

5. Let $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ be such that

- f is continuous on $[0,T] \times \mathbf{R}$,
- $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ are bounded on $[0,T] \times \mathbf{R}$.

Consider the solution $x : [0,T] \to \mathbf{R}$ of the differential equation

$$\frac{dx}{dt} = f(t,x),$$
$$x(0) = \alpha.$$

Split the interval $[0,T]$ into subintervals of size $h > 0$ so that $0 = t_0 < t_1 < \cdots < t_n = T$ with $t_i = ih$ are the grid points. Consider the Euler approximation of x on the grid points :

$$\tilde{x}(t_i) = \tilde{x}(t_{i-1}) + f\left(t_{i-1}, \tilde{x}(t_{i-1})\right), \qquad 1 \le i \le n$$
$$\tilde{x}(0) = \alpha.$$

Prove in detail that $\sup_i |x(t_i) - \tilde{x}(t_i)| = O(h)$ as $h \to 0$. **[7 marks]**